STABILITY LOSS IN THE CYLINDRICAL EQUILIBRIUM STATE OF A ROTATING LIQUID

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Loss of stability of the equilibrium state of a capillary liquid was defined in [1]. The present study will determine an upper limit for loss of stability in the circular cylinder equilibrium state of a rotating liquid confined between two parallel plates. The question of stability of this particular state was considered in [2], and the problem of branching was considered in [3-6]. To obtain an estimate of stability loss conditions for a circular column, we will consider all axisymmetric and planar forms of equilibrium for a liquid enclosed between parallel plates which rotate together with the liquid as a solid body, about an axis normal to the plates. The question of the limits of existence of the spatial forms (observed in [5]) and their effect on stability loss remains open. It would be interesting to study the problem with consideration of possible breakoff of liquid mass from the rotating column.

1. Between two parallel plates separated by a distance L there is enclosed a weightless viscous liquid with surface tension coefficient σ, density ρ, and volume $\pi r_0^2 L$. The liquid together with the plates rotates as a solid body with constant angular velocity ω about an axis normal to the plane of the plates. The center of mass of the liquid is located on the axis of rotation and the wetting angle is equal to $\pi/2$.

We introduce dimensionless variables by choosing the quantities r_0, ωr_0, $\rho\omega^2 r_0^2$ as scale factors for length, velocity, and pressure. Now let η, α, z be a rotating cylindrical coordinate system rigidly fixed to the plates. The z axis is directed along the axis of rotation, $z = 0$ and $z = l = L/r_0$ being the equations of the plate planes. Liquid equilibrium with respect to this coordinate system will be termed equilibrium of the rotating liquid.

For all values of the dimensionless parameter $\beta = \rho\omega^2 r_0^3/\sigma$ one of the possible forms of rotating liquid equilibrium is a circular cylindrical surface of radius $\eta \equiv 1$ [2].

The axisymmetric equilibrium surface is characterized by the line Γ along which it intersects the semiplane α = const. We will consider axisymmetric forms for which in motion along Γ from the plate $z = 0$ to the plate $z = l$ the distance from the axis of rotation changes monotonically. Such forms will be termed simple. In the case of monotonic increase simple equilibrium forms of the form $z = Z(\eta)$ are defined by the equation [2]

$$2H = (\beta/2)\eta^2 + c, \tag{1.1}$$

where

$$H = \frac{1}{2\eta}\left(\frac{\eta Z'}{(1+Z'^2)^{1/2}}\right)'. \tag{1.2}$$

To this we add the condition (wetting angle equal to $\pi/2$)

$$Z'(\eta_0) = Z'(\eta_1) = \infty, \tag{1.3}$$

the condition of conservation of liquid volume

$$\int\limits_{\eta_0}^{\eta_1} \eta^2 Z' d\eta = l, \tag{1.4}$$

and the equation

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$$\int_{\eta_0}^{\eta_1} Z' d\eta = l, \tag{1.5}$$

which reflects the fact that the distance between the plates is equal to l. Here and below a prime denotes differentiation with respect to η; c is an unknown constant; and η_0 and η_1 are the smallest and largest distances from the equilibrium surface to the axis of rotation.

Substitution of Eq. (1.2) in Eq. (1.1) followed by integration gives

$$\frac{\eta Z'}{(1+Z'^2)^{1/2}} = \frac{\beta}{8}\eta^4 + \frac{c}{2}\eta^2 + c_1 \tag{1.6}$$

(where c_1 is an integration constant).

Equations (1.3)-(1.6) define a two-parameter family of simple axisymmetric forms to the accuracy of the transformation $z = l - z$.

2. To study the properties of the axisymmetric forms we choose as independent parameters the following:

$$\theta = \eta_0/\eta_1, \quad b = \beta(1+\theta)\eta_1^3/8. \tag{2.1}$$

From the limit equations obtained from Eq. (1.6) with consideration of Eq. (1.3), we find expressions for the constants c and c_1 in terms of the parameters θ, b, η_1:

$$\begin{gathered} c = 2[1 - b(1+\theta^2)]/[\eta_1(1+\theta)], \\ c_1 = \theta\eta_1(1+b\theta^2)/(1+\theta). \end{gathered} \tag{2.2}$$

We substitute these expressions in Eq. (1.6) and solve the latter for Z'. In the equation thus obtained as well as in Eqs. (1.4), (1.5) we transform to the new variables $x = z/\eta_1$, $r = \eta/\eta_1$. As a result, from Eq. (1.6), introducing the notation

$$u(r,\theta,b) = \frac{r^2+\theta-b(1-r^2)(r^2-\theta^2)}{\sqrt{(1+r)(r+\theta)(1+2b(r^2+\theta)-b^2(1-r^2)(r^2-\theta^2))}}, \tag{2.3}$$

we obtain

$$\frac{dx}{dr} = \frac{u(r,\theta,b)}{\sqrt{(1-r)(r-\theta)}}, \tag{2.4}$$

from which it follows that

$$x = X(r,\theta,b) = \int_\theta^r \frac{u(\tau,\theta,b)\,d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}, \tag{2.5}$$

and from Eqs. (1.4), (1.5) we find the dependence of η_1 and l on the parameters θ, b

$$\eta_1 = \left[\int_\theta^1 \frac{u(\tau,\theta,b)\,d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}\right]^{1/2} \Bigg/ \left[\int_\theta^1 \frac{\tau^2 u(\tau,\theta,b)\,d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}\right]^{1/2}; \tag{2.6}$$

$$l = F(\theta,b) = \left[\int_\theta^1 \frac{u(\tau,\theta,b)\,d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}\right]^{3/2} \Bigg/ \left[\int_\theta^1 \frac{\tau^2 u(\tau,\theta,b)\,d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}\right]^{1/2}. \tag{2.7}$$

From the specified values of θ and b the equilibrium form is defined parametrically with the aid of Eqs. (2.3), (2.5), (2.6) and the formulas

$$z = \eta_1 X(r,\ \theta,\ b),\ \eta = \eta_1 r,\ \theta \leqslant r \leqslant 1.$$

Figure 1 shows the function $F(\theta, b)$, calculated with Eq. (2.7) by computer. Simple axisymmetric equilibrium forms exist only for $l < \pi$.

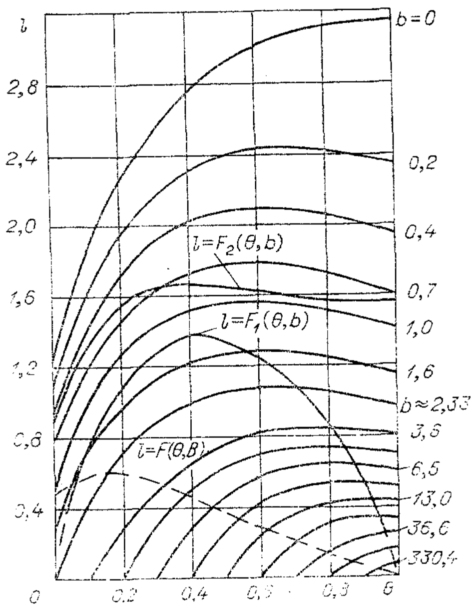

Fig. 1

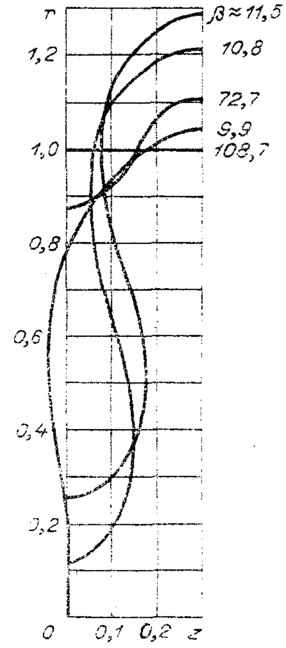

Fig. 2

The curve corresponding to the value $b = 0$ corresponds to equilibrium forms of the liquid at rest. Such forms exist if $l \geqslant F(0, 0) = \sqrt{1.5}$. If $\theta = 0$ $(0 < l < \sqrt{1.5})$, then $\lim\limits_{r\to 0}\frac{dX}{dr} = 0$ and the boundary condition on the plate $z = 0$ is disrupted. In this degenerate case we obtain equilibrium figures of a rotating liquid droplet, pendant on the plate $z = l$. At $b = 0$ the droplet comes to rest, and its surface has the form of a hemisphere of radius $l = \sqrt{1.5}$. With consideration of the fact that the problem of equilibrium forms is invariant relative to mirror reflection in the plane $z = l$, it can be said that at $\theta = 0$ equilibrium figures of an isolated rotating droplet are obtained. Such figures have been studied previously by many authors (see the bibliography of [1]), and in particular, their stability was considered in [7].

The dashed line of Fig. 1 depicts the curve $l = F(\theta, B)$, where $B = (1+\theta)^{-1} \times (1 - \sqrt{\theta})^{-2}$. Forms for which the point $F(\theta, b)$ lies above this curve are projectable unambiguously on the z axis, while in the opposite case their projections are ambiguous. This follows from Eq. (2.4) and the following property of the function $u(r, \theta, b)$, valid for all values of $\theta \in [0, 1)$: for $b < B$ for all $r \in [\theta, 1]$ the function $u(r, \theta, b) > 0$ (with the exception of the case $r = \theta = 0$, where $dX/dr = 0$), while for $b > B$ there exists an interval $\Delta \in (0, 1)$, such that for all r in Δ the function $u(r, \theta, b) < 0$. If $\theta = 0$, then $B = 1$, which agrees with the result of [7].

As $\theta \to 1$ the equilibrium figures tend to a circular cylindrical surface. From Eqs. (2.6), (2.7), after change of the integration variable $\tau = (1+\theta)/2 + (1-\theta)t/2$ we find that at $\theta = \theta^* = 1$ the parameter $\eta_1 \equiv 1$, and $l = \pi/\sqrt{1+4b}$; hence with consideration of Eq. (2.1) we find

$$\beta^* = 4b^* = (\pi/l)^2 - 1, \tag{2.8}$$

coinciding with the condition for branching of a circular cylindrical surface found in [4, 6].

At $l = 0$ the problem of branching of the circular cylindrical equilibrium state has no meaning. The system (1.4), (1.5) is nonsimultaneous in this case. However if we change condition (1.4), for example, by equating the liquid volume in the volume of a sphere of unit radius $\left(\int\limits_{\eta_0}^{\eta_1} \eta^2 Z' d\eta = 4/3\right)$, then at $l = 0$ Eqs. (1.3)-(1.6) will define a single parameter family of toroidal equilibrium figures. It is known [1] that all such figures are unstable. The curve $l = F(\theta, b_0)$, where $b_0 = 2.32912$, commences at the origin. The value of b_0 was calculated by Newton's method from the equation $\int_0^1 [u(\tau, 0, b)/\sqrt{\tau(1-\tau)}]\, d\tau = 0$. At $b = b_0$ rotating droplet equilibrium figures intersecting the axis of rotation cease to exist, and beginning with this value, toroidal figures exist. There has been discussion on the definition of the critical value b_0 (see [1, 8]), which was resolved in [7]. The value of b_0 determined in the present study coincides with that obtained in [7] and later confirmed in [8].

For $b > b_0$ the function $F(\theta, b)$ was constructed for b_i values such that at $l = 0$ and $b = b_i$ the parameter $\theta = \theta_i = 0.1i$ ($i = 1, 3, \ldots, 9$). The numbers b_i were calculated by Newton's method from the equation $\int_{\theta_i}^1 [u(\tau, \theta_i, b)/\sqrt{(1-\tau)(\tau-\theta_i)}\, d\tau = 0$ (for $\theta_i = 0.05i$, $i = 1, 2, \ldots, 19$), while for the initial approximation to b_i the value of b_{i-1} obtained in the preceding step was used.

Let $k \geqslant 2$ be an integer. We denote by Γ_i ($i = 0, 1, \ldots, k-1$) the segment of the arc of meridional section Γ, included between the planes $z = il/k$ and $z = (i+1)l/k$. We will say that an equilibrium form Γ has a multiplicity of k, if Γ_0 is a simple curve, and for every $i \geqslant 1$ the segment Γ_i can be obtained from Γ_{i-1} by mirror reflection in the plane $z = il/k$.

The branch of axisymmetric forms with multiplicity k branches from the cylindrical state at values $\beta_k = k^2(\pi/l)^2 - 1$.

Any axisymmetric equilibrium form is either simple or multiple. In fact, if two simple equilibrium surfaces are "extensions" of each other, forming one axisymmetric form, then each of the forms is uniquely defined by three parameters: η_0, η_1 and $\beta(\eta_0, \eta_1', \beta)$, where η_0 is the value of η in the "contact" plane, and η_1 and η_1' are the values on the solid planes. But in view of the continuity of the mean curvature, there follows from Eqs. (1.1), (2.1), (2.2) the equation $2/(\eta_0 + \eta_1) - \beta(\eta_0^2 + \eta_1^2)/2 = 2/(\eta_0 + \eta_1') - \beta(\eta_0^2 + \eta_1'^2)/2$, which is valid only for $\eta_1 = \eta_1'$.

At $l > \pi$ there exist only multiple forms.

We will fix l and find the function $b(\theta)$ by linearizing Eq. (2.7) in the vicinity of the critical values θ^*, b^*:

$$F(\theta^*, b^*) - l + \frac{\partial F}{\partial \theta}(\theta^*, b^*)(\theta - \theta^*) + \frac{\partial F}{\partial b}(\theta^*, b^*)(b - b^*) = 0.$$

Hence, considering that $F(\theta^*, b^*) = l$, we find

$$b = b^* + (\theta^* - \theta)\frac{\partial F}{\partial \theta}(\theta^*, b^*)\Big/\frac{\partial F}{\partial b}(\theta^*, b^*).$$

Setting $\theta = \theta_h = \theta^* - h$ (in numerical computation the value of h was chosen equal to 10^{-2}), we obtain an approximate solution b_h to Eq. (2.7). Linearizing the equation $F(\theta_h, b) - l = 0$ in the vicinity of $b = b_n$, we find a formula for the refinement of the root b_h

$$b = b_h + (l - F(\theta_h, b_h))\Big/\frac{\partial F}{\partial b}(\theta_h, b_h).$$

With this formula the value of b_h is refined to the point where the difference $|b - b_h|$ becomes less than 10^{-6}. For the refined value of b_h and $\theta = \theta_h$ Eq. (2.7) is accurate to six decimal places. In this manner, moving with a step h in θ in the direction of lower θ, we find the function $b(\theta)$. At each step values of $\eta_1(\theta)$ and $\beta(\theta)$ (Eqs. (2.6), (2.1)) are also calculated. Expressions for $\partial F/\partial\theta$, $\partial F/\partial b$ will not be presented because of their cumbersomeness.

Figure 2 shows meridional sections of equilibrium surfaces for $l = 0.3$ and several values of the parameter θ (which uniquely defines the value of β). It is interesting that at small β the equilibrium forms intersect the plane $z = 0$. Such forms are physically realizable if the dimensions of the plate $z = 0$ are sufficiently small. At $\beta = 9.93$ ($\theta = 0$) the equilibrium form degenerates into a rotating droplet with some concavity at its pole.

3. We will consider the stability of axisymmetric equilibrium states in the sense of Lyapunov and Rumyantsev [1]. Let Γ be a simple equilibrium form, s the ratio of the arc length Γ, measured from the plate $z = 0$ to the farthest removal η_1 of curve Γ from the axis of rotation, and $N(s, \alpha)$ the normal component of the free surface perturbation, referred to η_1. Commencing from the principle of minimum potential energy [1, 8], the question of

stability of the axisymmetric form Γ can be reduced to the problem of eigenvalues of the following linear boundary problem for the function $N(s, \alpha)$:

$$-\frac{1}{r}\frac{\partial}{\partial s}\left(r\frac{\partial N}{\partial s}\right)-\frac{1}{r^2}\frac{\partial^2 N}{\partial \alpha^2}+aN+\mu=\lambda N \quad \begin{pmatrix}0\leqslant s\leqslant s_1,\\ 0\leqslant\alpha\leqslant 2\pi\end{pmatrix}; \tag{3.1}$$

$$\frac{\partial N}{\partial s}=0 \quad (s=0),\quad \frac{\partial N}{\partial s}=0 \quad (s=s_1); \tag{3.2}$$

$$\int_0^{s_1}\int_0^{2\pi} N r\,ds\,d\alpha=0,\quad \int_0^{s_1}\int_0^{2\pi} Nr\cos\alpha\,ds\,d\alpha=\int_0^{s_1}\int_0^{2\pi} Nr\sin\alpha\,ds\,d\alpha=0. \tag{3.3}$$

Here s_1 is the value of s at the point of intersection of Γ with the plane $z = l$. With specified values of θ and b the value of s_1 is given by

$$s_1=(1+\theta)\int_\theta^1\frac{rG(r,\theta,b)\,dr}{\sqrt{(1-r^2)(r^2-\theta^2)}}, \tag{3.4}$$

where

$$G(r,\theta,b)=1/\sqrt{1+2b(r^2+\theta)-b^2(1-r^2)(r^2-\theta^2)}. \tag{3.5}$$

The function a is expressed in terms of the mean curvature H and the Gaussian curvature $K = Z'Z''/[(1 + Z'^2)^2]$ with the formula

$$\eta_1^2 a=-\frac{\partial H}{\partial n}-4H^2+2K,\quad \frac{\partial}{\partial n}=\mathbf{n}\cdot\nabla, \tag{3.6}$$

where $\mathbf{n}$ is the unit normal vector to the surface, directed into the liquid filled region; the first equation of Eqs. (3.3) expresses the conservation of liquid volume, while the second implies the admissability of only those perturbations which leave the center of the liquid mass on the axis of rotation; the dependence of r on s is given by

$$s=J(r)=(1+\theta)\int_\theta^r\frac{\tau G(\tau,\theta,b)\,d\tau}{\sqrt{(1-\tau^2)(\tau^2-\theta^2)}}. \tag{3.7}$$

The eigenvalues of the problem of Eqs. (3.1)-(3.7) are real. If the smallest eigenvalue λ_* is positive, then the corresponding equilibrium state of the viscous liquid is stable, while if λ_* is negative, then it is unstable.

Representing the function $N(s, \alpha)$ in the form of a series $N = \varphi_0(s)+\sum_{m=1}^{\infty}[\varphi_m(s)\cos m\alpha+\psi_m(s)\sin m\alpha]$, it can be shown that $\lambda_* = \min(\lambda_0, \lambda_1, \lambda_2)$, where λ_0 is the smallest of the eigenvalues of the problem

$$\frac{1}{r}\frac{d}{ds}\left(r\frac{d\varphi_0}{ds}\right)-a\varphi_0-\mu+\lambda\varphi_0=0 \quad (0\leqslant s\leqslant s_1); \tag{3.8}$$

$$\frac{d\varphi_0}{ds}=0 \quad (s=0,\ s=s_1),\quad \int_0^{s_1}\varphi_0 r\,ds=0, \tag{3.9}$$

and the numbers λ_m ($m = 1, 2$) are defined similarly for the problem

$$\begin{gathered}\frac{1}{r}\frac{d}{ds}\left(r\frac{d\varphi_m}{ds}\right)-\left(a+\frac{m}{r^2}\right)\varphi_m+\lambda\varphi_m=0 \quad (0\leqslant s\leqslant s_1),\\ \frac{d\varphi_m}{ds}=0 \quad (s=0,\ s=s_1),\quad \int_0^{s_1}\varphi_1 r^2 ds=0 \quad (m=1,2).\end{gathered} \tag{3.10}$$

After simple transformations we obtain from Eq. (3.6)

$$a = -\frac{2}{(1+\theta)^2}\left[4bP + \left(\frac{P}{r^2} - Q\right)^2 - Q^2\right], \tag{3.11}$$

where

$$P = r^2 + \theta - b(1 - r^2)(r^2 - \theta^2);$$
$$Q = 2br^2 + 1 - b(1 + \theta^2). \tag{3.12}$$

The eigenvalues of Eqs. (3.8)-(3.10) are calculated by the Galerkin–Ritts method [9]. Let $\{y_k(s)\}_{k=1,2,\ldots}$ be the complete system of functions satisfying condition (3.9):

$$y_1 = 2s^3 - 3s^2 s_1 - \frac{J_1}{J_0},\quad y_k = s^2(s - s_1)^k - \frac{J_k}{J_0} \quad (k = 2, 3, \ldots),$$

$$J_0 = \int_0^{s_1} r\,ds = (1+\theta)\int_\theta^1 \frac{r^2 G(r,\theta,b)\,dr}{\sqrt{(1-r^2)(r^2-\theta^2)}},$$

$$J_1 = \int_0^{s_1}(2s^3 - 3s^2 s_1)\,r\,ds = (1+\theta)\int_\theta^1 \frac{r^2\,[2J^3(r) - 3J(1)J^2(r)]\,G(r,\theta,b)\,dr}{\sqrt{(1-r^2)(r^2-\theta^2)}}, \tag{3.13}$$

$$J_k = \int_0^{s_1} s^2(s - s_1)^2 r\,ds = (1+\theta)\int_0^1 \frac{r^2 J^2(r)\,[J(r) - J(1)]^2 G(r,\theta,b)\,dr}{\sqrt{(1-{}^2)(r^2-\theta^2)}}.$$

Expanding $\varphi_0(s)$ in the form of a series $\varphi_0 = \sum_{k=1}^{n} d_k y_k(s)$ and striving for orthogonality of the left side of Eq. (3.8) in the functions $y_1, \ldots, y_n$, we arrive at the system of equations

$$\sum_{p=1}^{n}(\alpha_{p,q} + \lambda\gamma_{p,q})\,d_p = 0 \quad (q = 1, 2, \ldots, n), \tag{3.14}$$

where

$$\alpha_{p,q} = -(1+\theta)\int_\theta^1\left(\frac{dy_p}{ds}\frac{dy_q}{ds} + a y_p y_q\right)\frac{r^2 G(r,\theta,b)\,dr}{\sqrt{(1-r^2)(r^2-\theta^2)}}; \tag{3.15}$$

$$\gamma_{p,q} = (1+\theta)\int_\theta^1 y_p y_q r^2\,\frac{G(r,\theta,b)\,dr}{\sqrt{(1-r^2)(r^2-\theta^2)}}.$$

The functions y_k and their derivatives within the integrand of Eq. (3.15) should be expressed in terms of r by means of Eqs. (3.7), (3.13). Setting equal to zero the determinant of system (3.14), we obtain

$$\begin{vmatrix} \alpha_{1,1} + \lambda\gamma_{1,1} \ldots \alpha_{n,1} + \lambda\gamma_{n,1} \\ \cdots \\ \alpha_{n,1} + \lambda\gamma_{n,1} \ldots \alpha_{n,n} + \lambda\gamma_{n,n} \end{vmatrix} = 0, \tag{3.16}$$

whose smallest root gives the approximate value of λ_0 (approximation with excess). Equations for finding λ_1 and λ_2 are found similarly.

In numerical calculation the set of parameters θ, b are chosen in the following manner. Initially, for each fixed value of $b_i = 0.04i$ ($i = 0.1, \ldots, 58$) the parameter θ is varied in steps of 0.05 from 0 to 0.95. Then values of b_i ($i = 1, 2, \ldots, 18$), are fixed such that the curve $l = F(\theta, b_i)$ intersects the axis $l = 0$ at the value $\theta_i = 0.05i$; at each b_i value in this case the parameter θ is varied in the same steps as before over the range $[\theta_i, 0.95]$. For each pair (θ, b) the coefficients α_{pq} and γ_{pq} are calculated with Eq. (3.15), after which Eq. (3.16) with number of coordinate functions 1, 2, ..., n is used to find the approximations $\lambda_0^{(1)}, \lambda_0^{(2)}, \ldots, \lambda_0^{(n)}$ (correspondingly $\lambda_m^{(1)}, \lambda_m^{(2)}, \ldots, \lambda_m^{(n)}$, m = 1, 2). To prove the instability of all simple forms (and thus, all multiple forms) it is sufficient to calculate only the value of $\lambda^{(1)}_{\circ}$ at $\theta < 0.95$ and the numbers $\lambda^{(1)}_{\circ}$, $\lambda^{(2)}_{\circ}$, $\lambda^{(3)}_{\circ}$ for $\theta = 0.95$; if $\theta < 0.95$ then for all sets of parameters (θ, b) the values of $\lambda^{(1)}_{\circ}$ are negative, while at $\theta = 0.95$ the numbers $\lambda^{(3)}_{\circ}$ are negative (and close to zero).

For $\theta = 1$, $r \equiv 1$ Eqs. (3.8)-(3.12) comprise the problem of stability of a cylindrical equilibrium state [2]. The critical values of the parameter b, defined by Eq. (2.8), correspond to the eigennumbers $\lambda_0 = 0$.

Numerical calculations reveal that to find the values of λ_1 and λ_2 to an accuracy of two decimal places, it is sufficient to limit the number of coordinate functions to n = 4. For n = 3.4 Eq. (3.16) was solved by Newton's method, with the initial approximation to $\lambda_m^{(n)}$ being taken in the form of the root $\lambda_m^{(n-1)} > \lambda_m^{(n)}$. For b = 0 the values of λ_1, $\lambda_2 > 0$. For fixed θ with increase in b the numbers λ_1 and λ_2 decrease monotonically. The curve $l = F_1(\theta, b)$ (see Fig. 1) was constructed with those values of θ and b, for which $\lambda_1 = 0$, and on the curve $l = F_2(\theta, b)$ the number λ_2 goes to zero. In particular, for $l = F_2(1, 3/4) = \pi/2$ the value $\lambda_2 = 0$ [2], and, according to the results of [5], at these parameter values, aside from axisymmetric and plane equilibrium forms, there are also spatial equilibrium forms branching from the cylindrical state. It is completely possible for branching of the axisymmetric states to occur at $l = F_2(\theta, b)$ (and also possible at $l = F_1(\theta, b)$).

Results of the study of stability of a rotating viscous droplet are presented in [1]. The isolated viscous droplet is stable for $0 \leqslant b \leqslant b_{10} \approx 0.4587$. Upon transition through the value b_{10} stability is lost relative to second harmonic perturbations. According to the results of the present study a droplet confined between parallel plates ($\theta = 0$) is unstable (relative to axisymmetric perturbations) for all values of the parameter b. The difference in the results is explained by the fact that, for perturbations as small as desired satisfying the wetting conditions on the plates rotating constrained at a specified velocity, there occurs an increase in system energy leading to the increase of the perturbations. It is curious that a droplet at rest in contact with parallel plates is also unstable.

4. The cylindrical equilibrium surface is characterized by a line of intersection with the plane z = const. We will consider nonaxisymmetric cylindrical figures, the normal sections of which have n-fold symmetry relative to the z axis (n = 2, 3, ...). We will refer to such figures as planar equilibrium forms of multiplicity 2n. A planar form of multiplicity 2n is defined (to the accuracy of rotation about the z axis) if the function $\alpha = A_1(\eta)$ is known, which specifies the simple segment of its normal section.

The function $A_1(\eta)$ satisfies Eq. (1.1) in which now

$$H = \frac{1}{2\eta}\left(\frac{A_1'\eta^2}{(\eta^2 A_1'^2 + 1)^{1/2}}\right)', \tag{4.1}$$

and also the conditions of conservation of volume,

$$\int\limits_{\eta_0}^{\eta_1} \eta^2 A_1' d\eta = \frac{\pi}{n}, \tag{4.2}$$

symmetry,

$$A_1'(\eta_0) = A_1'(\eta_1) = \infty \tag{4.3}$$

and periodicity

$$\int\limits_{\eta_0}^{\eta_1} A_1' d\eta = \frac{\pi}{n}. \tag{4.4}$$

For right cylindrical forms the boundary conditions on the plates are fulfilled automatically. From Eq. (1.1) with consideration of Eq. (4.1) after integration we obtain

$$\frac{A_1'\eta^2}{(\eta^2 A_1'^2 + 1)^{1/2}} = \frac{\beta}{8}\eta^4 + \frac{c}{2}\eta^2 + c_1. \tag{4.5}$$

For definiteness we will add to system (4.2)-(4.5) the condition $A_1(\eta_0) = 0$. We transform to the variable $r = \eta/\eta_2$ and introduce the parameters of Eq. (2.1). Then to find the normal sections we obtain the parametric equations

$$\alpha = A(r,\ \theta,\ b),\ \eta = R(\theta,\ b)r,\ b = b(\theta),$$
$$0 < \theta < 1,\ \theta \leqslant r \leqslant 1,$$

where

$$A(r,\theta,b) = \int\limits_{\theta}^{r} \frac{u(\tau,\theta,b)\,d\tau}{\tau\sqrt{(1-\tau)(\tau-\theta)}};\ R(\theta,b) = \left[\frac{\pi}{n\int\limits_{\theta}^{1}\frac{\tau u(\tau,\theta,b)\,d\tau}{\sqrt{(1-\tau)(\tau-\theta)}}}\right]^{1/2}, \tag{4.6}$$

and the function $b(\theta)$ is determined numerically (as in section 2 for the case of axisymmetric forms) from the equation

$$\int\limits_{\theta}^{1} \frac{u(\tau,\theta,b)\,d\tau}{\tau\sqrt{(1-\tau)(\tau-\theta)}} = \frac{\pi}{n}, \tag{4.7}$$

to which must be added the bifurcation condition

$$4b_{(n)}^{*} = \beta_{(n)}^{*} = n^2 - 1 \quad (n = 2, 3, \ldots). \tag{4.8}$$

This condition is obtained from Eq. (4.7), if we transform to the limit as $\theta \to 1$.

The family of cylindrical equilibrium surfaces is characterized by two independent parameters. For these parameters we choose the distance between the plates l and the parameter θ, which with the aid of Eqs. (2.1), (4.6)-(4.8) will be used to define the value of the dimensionless angular velocity β.

Plane figures were studied in the vicinity of the critical values $\beta_{(n)}^{*}$ in [3]. Expanding the solution of the general problem of cylindrical forms continuously deviating from a circular cylinder and applying the normal method of expansion of the solution in powers of the small deviation amplitude, the authors of [3] found bifurcation points (4.8), constructed the equilibrium figures in the vicinity of these points, and also obtained the relationship between the amplitude deviation of the normal section of these figures and the angular velocity of rotation. The functions found by them are shown as the curves γ_2 and γ_3 of Fig. 3. The branches γ_2' and γ_3' were constructed from the exact formulas (2.1), (4.6)-(4.8) with numerical calculations. In the vicinity of the bifurcation points the curves γ_n and γ_n' coincide. The difference in their further behavior is explained by the fact that the approximate solution obtained in [3] loses force.

Figure 4 shows the complete path of the bifurcation curves for n = 2, 3, ..., 6.

Cylindrical forms, the normal sections of which have second order symmetry about the z axis, exist to the value $\theta = 0$ ($\beta \approx 0.79$). At $\theta = 0$ the cylindrical surface intersects the axis of rotation and such a surface is devoid of physical meaning.

With order of symmetry $n > 2$ the cylindrical forms cease to exist even at some $\theta(n) > 0$. Numbers $\beta_{(n)}$ corresponding to these $\theta_{(n)}$ are shown in Fig. 4. When the value $\beta_{(n)}$ is reached, self-intersections form on the cylindrical surface.

Figure 5 depicts the form of normal cylindrical figures for n = 3 and several values of the parameter β.

At n = 1 the center of mass of the plane figures does not lie on the axis of rotation. The problem of Eqs. (4.2)-(4.5) for such figures will not be considered here.

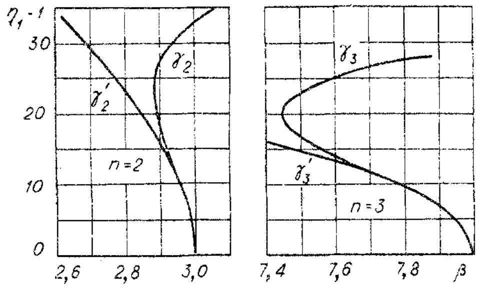

Fig. 3

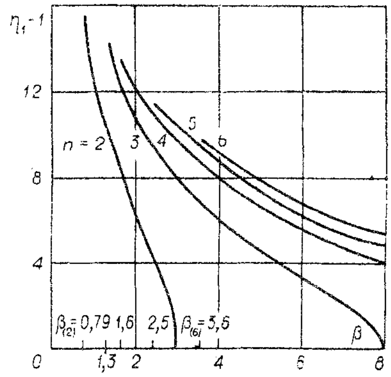

Fig. 4

Just as in the case of axisymmetric forms, it can be shown that no cylindrical forms lacking symmetry relative to the axis of rotation are stable.

5. We will study the stability of the cylindrical figures. For an equilibrium figure with even order symmetry we will consider plane perturbations

$$\varphi(s) = c_n(\cos(2\pi ns/s_1) + \sin(2\pi ns/s_1)), \tag{5.1}$$

where s is the arc length of the normal section referred to η_1. The value of s varies over the range $0 \leqslant s \leqslant s_1$, the points $s = 0$ and $s = s_1$ corresponding to the values $\alpha = 0$ and $\alpha = 2\pi$. The arbitrary constant c_n characterizes the closeness of the perturbed surface to the equilibrium surface considered. The function φ satisfies the unambiguity conditions

$$\varphi(0) = \varphi(s_1),\ d\varphi/ds(0) = d\varphi/ds(s_1). \tag{5.2}$$

The perturbations of Eq. (5.1) maintain the liquid volume

$$\int_0^{s_1} \varphi ds = 0 \tag{5.3}$$

and do not displace its center of mass from the axis of rotation

$$\int_0^{s_1} \varphi r \cos\frac{2\pi s}{s_1}\,ds = \int_0^{s_1} \varphi r \sin\frac{2\pi s}{s_1}\,ds = 0. \tag{5.4}$$

The sign of the second variation of the potential energy in the perturbations φ is determined (see [1]) from the sign of the integral

$$I_0 = \int_0^{s_1} \left[\left(\frac{d\varphi}{ds}\right)^2 + a\varphi^2\right] ds. \tag{5.5}$$

The function a is calculated from Eq. (3.6), and for cylindrical forms appears as

$$a = -4(2bP + Q^2)/(1+\theta)^2.$$

Substituting the functions a, φ, dφ/ds in Eq. (5.5), we obtain

$$I_0 = c_n^2\left[\frac{4\pi^2 n^2}{s_1} + 2n\int_0^{s_1/2n} a\,ds\right] = \frac{4c_n^2 n}{(1+\theta) I_1}[\pi^2 - I_1 I_2],$$

where

$$I_1 = \int_{\theta^2}^1 \frac{G(\tau,\theta,b)\,d\tau}{\sqrt{(1-\tau)(\tau-\theta^2)}};\quad I_2 = \int_{\theta^2}^1 \frac{(2bP+Q^2)\,G(\tau,\theta,b)\,d\tau}{\sqrt{(1-\tau)(\tau-\theta^2)}}, \tag{5.6}$$

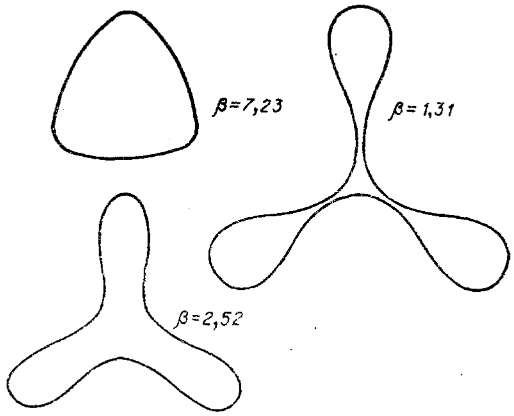

Fig. 5

the functions $P(\tau)$, $Q(\tau)$, $G(\tau, \theta, b)$ are defined by Eqs. (3.12), (3.5) if we take $r^2 = \tau$ therein.

A plane figure with even order symmetry is unstable (in the Lyapunov—Rumyantsev sense), if for its parameter values θ, $b(\theta)$ the inequality

$$I_1 I_2 > \pi^2 \tag{5.7}$$

is fulfilled. At $\theta = 1$ the integral $I_1 = \pi/\sqrt{1 + 4b} = \pi/n$, and $I_2 = \pi\sqrt{1 + 4b} = \pi n$, so that for any n the product $I_1 I_2 = \pi^2$. Inasmuch as this product is independent of b, at the bi-furcation point the derivative "along the branch" $d(I_1/I_2)/d\theta|_{\theta=1}$ coincides with the partial derivative $\partial(I_1 I_2)/\partial\theta|_{\theta=1}$. With the aid of Eq. (5.6) we find that at $\theta = 1$ the derivative $\partial(I_1 I_2)/\partial\theta = 0$, while $d^2(I_1 I_2)/d\theta^2 = \partial^2(I_1 I_2)/\partial\theta^2 = 12\pi b^3/(1 + 4b)^4 > 0$. From this it follows that for each branch Eq. (5.7) is satisfied in the vicinity of the bifurcation point. In other words, all plane figures with even order symmetry which branch from a circular cylindrical state at values $\beta^*_{(n)} = n^2 - 1$ are unstable in the vicinity of these values.

Along the branches with n = 2.4 the product $I_1 I_2$ was calculated with a computer. It was shown that with decrease in the parameter θ (or β) this product increases monotonically.

We note further that for a circular rotating column $a = -\beta - 1$, $s_1 = 2\pi$, $I_0 = 2\pi c_n^2(n^2 - 1 - \beta)$. Thus with increase in β the second variation of the potential energy in the perturbations of Eq. (5.1) decreases and changes sign to negative upon transition through the bifurcation value $\beta^*_{(n)}$.

For the equilibrium states with odd order symmetry n ($n \geqslant 3$) a perturbation of the form

$$\varphi = c_n \left(I_{(n-1)} \cos \frac{2\pi n s}{s_1} - I_{(n)} \cos \frac{2\pi (n-1) s}{s_1} \right), \tag{5.8}$$

where $I_{(\cdot)} = \int\limits_0^{s_1} r \cos \frac{2\pi (\cdot) s}{s_1} \cos \frac{2\pi s}{s_1} ds$, satisfies the necessary conditions (5.2)-(5.4). For $\theta = 1$ the integrals $I_{(n)}$ and $I_{(n-1)}$ are equal to zero, and consequently, $I_0 = 0$. We will show that nontrivial equilibrium figures with parameters θ and b close to the bifurcation values are unstable in the vicinity of the bifurcation points Eq. (4.8). For the perturbations of Eq. (5.8) we represent the integral I_0 in the form

$$I_0 = \frac{2nc_n^2}{(1+\theta) I_1} \left[I_{(n-1)}^2 (\pi^2 - I_1 I_2) + I_{(n)}^2 \left(\frac{\pi^2 (n-1)^2}{n^2} - I_1 I_2 \right) \right] -$$
$$- \frac{2c_n^2}{(1+\theta)^2} \left\{ I_{(n-1)}^2 \int\limits_0^{s_1} (2bP + Q^2) \cos \frac{4\pi n s}{s_1} ds + I_{(n)}^2 \int\limits_0^{s_1} (2bP + Q^2) \times \right.$$
$$\left. \times \cos \frac{4\pi (n-1) s}{s_1} ds - I_{(n-1)} I_{(n)} \int\limits_0^{s_1} (2bP + Q^2) \cos \frac{2\pi n s}{s_1} \cos \frac{2\pi (n-1) s}{s_1} ds \right\}.$$

It can be shown that at $\theta = 1$, along with the expressions in the squared and figured brackets the first derivatives of those same expressions with respect to θ go to zero, while for the expressions in figured brackets the second derivatives with respect to θ are also equal to zero. With consideration of these facts, we find that at $\theta = 1$ the derivative $dI_0/d\theta = \partial I_0/\partial\theta = 0$ and

$$d^2I_0/d\theta^2 = \partial^2 I_0/\partial\theta^2 = 2\pi^2 n I_1^{-1}\left[n^{-2}(n-1)^2 - 1\right](\partial I_{(n)}/\partial\theta)^2 = -2\pi^3(2n-1)n^{-2} < 0.$$

From this it follows that in the vicinity of the bifurcation points $I_0 < 0$, proving the instability of the branched equilibrium figures with odd order symmetry.

6. We define the potential energy of the rotating liquid by the formula [1]

$$U = \sigma\sum - \frac{1}{2}\omega^2 I, \tag{6.1}$$

where I is the moment of inertia; Σ is the area of the liquid free surface. For simple axisymmetric equilibrium states

$$\sum = 2\pi r_0^2\eta_1^2(1+\theta)\int_\theta^1 \frac{r^2G(r,\theta,b)\,dr}{\sqrt{(1-r^2)(r^2-\theta^2)}}, \quad I = 2\pi\rho r_0^5\eta_1^5\int_\theta^1 \frac{r^4u(r,\theta,b)\,dr}{\sqrt{(1-r)(r-\theta)}}.$$

Substituting these expressions in Eq. (6.1), we obtain

$$E_1 = \frac{U}{2\pi r_0^2\sigma} = \eta_1^2\left[(1+\theta)\int_\theta^1 \frac{r^2G(r,\theta,b)\,dr}{\sqrt{(1-r^2)(r^2-\theta^2)}} - \frac{b}{(1+\theta)}\int_\theta^1 \frac{r^4u(r,\theta,b)\,dr}{\sqrt{(1-r)(r-\theta)}}\right]. \tag{6.2}$$

For fixed l the function $b(\theta)$ can be defined by the numerical method described in Sec. 2. For an axisymmetric form of multiplicity k, characterized by the parameters θ, $b(\theta)$, the energy $E_k = kE_1$.

For planar equilibrium figures with n-fold symmetry relative to the axis of rotation

$$\sum = 2nl\eta_1 r_0^2(1+\theta)\int_\theta^1 \frac{rG(r,\theta,b)\,dr}{\sqrt{(1-r^2)(r^2-\theta^2)}}, \quad I = 2\rho nl\eta_1 r_0^5\int_\theta^1 \frac{r^3u(r,\theta,b)\,dr}{\sqrt{(1-r)(r-\theta)}}.$$

Hence, with the aid of Eq. (6.1) we find

$$E_2 = \frac{U}{2\pi r_0^2\sigma} = \frac{\eta_1 nl}{\pi}\left[(1+\theta)\int_\theta^1 \frac{rG(r,\theta,b)\,dr}{\sqrt{(1-r^2)(r^2-\theta^2)}} - \frac{b}{1+\theta}\int_\theta^1 \frac{r^3u(r,\theta,b)\,dr}{\sqrt{(1-r)(r-\theta)}}\right]. \tag{6.3}$$

For a specified n the function $b(\theta)$ is determined numerically with Eqs. (4.7) and (4.8).

For a circular cylindrical equilibrium state

$$E_0 = \frac{U}{2\pi r_0^2\sigma} = l\left(1 - \frac{\beta}{8}\right).$$

Using Eqs. (6.2) and (6.3), the energy of axisymmetric and cylindrical forms was calculated numerically for a number of values of the parameters l and n (steps of 0.01 in θ), and then for a given l (or n), using the value of $b(\theta)$ thus found, the parameter β was determined with the aid of Eqs. (2.1) and (2.6) (or Eqs. (2.1) and (4.6)). For each β together with the value of E_1 (or E_2) the energy E_0 of the circular cylindrical state was also calculated.

According to the results of [2], a circular cylindrical column of rotating viscous liquid is stable, if

$$\beta < \beta_* = \min\{\beta^*, \beta^*_{(2)}\} = \min\{(\pi/l)^2 - 1.3\}.$$

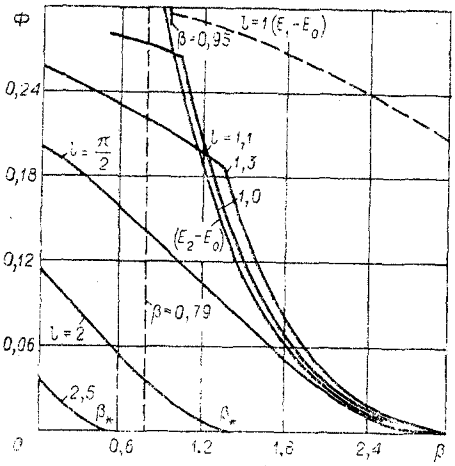

Fig. 6

For $\beta < \beta_*$ we introduce the function

$$\Phi(l, \beta) = \min_{i=1,2} \{E_i(l, \beta) - E_0(l, \beta)\}, \tag{6.4}$$

where the minimum is calculated over all axisymmetric (simple and multiple) and all plane equilibrium forms, existent at the values of l and β.

Equation (6.4) provides an upper limit for loss of stability of an equilibrium viscous liquid, characterized by parameters l and β. The function Φ is presented in Fig. 6 for certain l values.

For $l \geqslant \pi/2$ the cylindrical state is stable if $\beta < \beta_* \leqslant 3$. For such β values the lowest potential energy is found in simple axisymmetric forms, existing for $0 \leqslant \beta \leqslant \beta_*$. The function $\Phi(\beta)$ is presented in this case for the values $l = 2.5$; 2; $\pi/2$. For fixed l the function $\Phi(\beta)$ reaches its maximum value Φ_{max} at $\beta = 0$. If $l = \pi/2$, then $\Phi_{max} \approx 0.204$. With increase in l the value of Φ_{max} decreases. For $l = 2.5$ the value is 0.04.

For $l < \pi/2$ the critical value $\beta_* = 3$. If $\sqrt{1.5} \leqslant l < \pi/2$, then $\Phi(\beta)$ is defined for all $0 \leqslant \beta \leqslant 3$. Beginning at the value $\beta = 0$ up to some value $\beta_l \in (0.79;\ 3)$ the lowest potential energy is found in simple axisymmetric forms, and for $\beta > \beta_l$, in figures with fourfold symmetry. For $l < \sqrt{1.5}$ the function is defined only beginning with some value $\beta = \beta_l^0 > 0$, corresponding to a degenerate axisymmetric form $\Phi(\beta)$. For $l = 1.1$ the number $\beta_l^0 = 0.51$, while $\beta_l = 0.104$. For $l = 1$ the function $E_1(\beta)$ is defined only for $\beta > 0.95 > 0.79$. The curves $\Phi = E_1(1, \beta) - E_0(1, \beta)$, and $\Phi = E_2(1, \beta) - E_0(1, \beta)$ do not intersect. For $l \leqslant 1$ the function $\Phi(l, \beta) = E_2(l, \beta) - E_0(l, \beta)$, where $E_2(l, \beta)$ is the potential energy of a plane figure with $n = 2$ symmetry. For each $l \leqslant 1$ the maximum value of Φ_{max} is achieved at $\beta = 0.79$. For $l = 1$ this is equal to 0.34. The function $\Phi(\beta)$ itself is defined for $l \leqslant 1$ in the interval (0.79; 3).

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EFFECT OF INTERPHASE TANGENTIAL FORCES ON FLOW DEVELOPMENT IN A WEAKLY CONDUCTIVE LIQUID

A. I. Zhakin

UDC 533

Coulomb forces induce motion in a weakly conductive polarizable liquid by means of volume forces [1-5] and tangential surface stresses [2, 6]. While the former type of flow has a threshold character [1-5], the latter can develop in a vanishingly small electric field upon motion of surface charge over the free surface of the liquid [6]. The surface charge accumulation time on the free surface is of the order of magnitude of the free charge relaxation time $t_e = \varepsilon/\sigma$ [7]. If the problem characteristic time t_0 satisfies the inequality $t_0 \ll t_e$, then surface charge can be neglected and the major role will be played by polarization forces (for example, in problems involving stabilization of the free surface of a dielectric liquid by an electric field [2, 8, 9]). For $t_0 \geqslant t_e$ Coulomb surface forces cannot be ignored, and their consideration leads to the possibility of electroconvective flows.

In the present study the basic principles of thresholdless electroconvection will be considered, using the example of flow of a weakly conductive polarizable liquid under the action of surface forces produced by a special electrode geometry.

1. Formulation of the Problem. We will consider two incompressible viscous weakly conductive polarizable immiscible liquids, situated between two infinite horizontal electrodes and separated by a free surface S. We introduce a cartesian coordinate system as shown in Figs. 1, 2, and denote by Ω_i the region occupied by liquids, with $S_1 = (-\infty < x < \infty,\ z = h_1 + a \cos \omega x)$ being the upper curved electrode, and $S_2 = (-\infty < x < \infty,\ z = -h_2)$, the lower planar electrode. Here and below, the indices 1 and 2 refer to quantities defined in the regions Ω_1, Ω_2.

Liquid motion will be described by the electrohydrodynamics equations

$$\begin{gathered}\rho_i(\partial \mathbf{v}_i/\partial t + (\mathbf{v}_i\nabla)\mathbf{v}_i) = -\nabla p_i + \eta_i\Delta\mathbf{v}_i + q_i\mathbf{E}_i - \rho_i g\mathbf{e}_z,\\ \operatorname{div}\mathbf{v}_i = 0,\ \operatorname{div}\varepsilon_i\mathbf{E}_i = 4\pi q_i,\ \mathbf{E}_i = -\nabla\varphi_i,\\ \partial q_i/\partial t + \operatorname{div}\mathbf{j}_i = 0 \text{ on } \Omega_i,\end{gathered} \tag{1.1}$$

where ρ_i is the density; p_i is total pressure [8]: $\mathbf{j}_i = \sigma_i\mathbf{E}_i + q_i\mathbf{v}_i$ is current density; η_i, σ_i are constant dynamic viscosity and conductivity coefficients; q_i is volume charge density; φ_i is electric field potential; ε_i is dielectric permittivity; g is acceleration of gravity (i = 1, 2).

The boundary conditions for Eq. (1.1) follow from the conditions of adhesion, specification of potential on the electrodes, and kinematic, dynamic, and electrodynamic conditions on the free surface. They have the form [6, 8]

$$\begin{gathered}S_1 : \mathbf{v}_1 = 0,\ \varphi_1 = U = \text{const};\ S_2 : \mathbf{v}_2 = 0,\ \varphi_2 = 0;\\ S : \langle\mathbf{v}\rangle = 0,\ \partial f/\partial t = \mathbf{v}_1\cdot\mathbf{n}|\nabla f|^{1/2},\ \varphi_1 = \varphi_2,\\ \langle\varepsilon\mathbf{E}\cdot\mathbf{n}\rangle = 4\pi q_S,\ \partial q_S/\partial t + \operatorname{div}_S\mathbf{i} - Hq_S\mathbf{v}_1\cdot\mathbf{n} + \langle\mathbf{j}\mathbf{n}\rangle = 0,\end{gathered}$$

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